Math 1552

Section 8.8: Improper Integrals

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Today’s Learning Goals

- Be able to identify when an integral is improper (three cases)
- Rewrite an improper integral as a limit
- Understand the meaning of convergence and divergence as relating to integration
- Evaluate improper integrals

$$\int_a^b f(x)\,dx$$

Improper integrals

$$\int_0^{2\pi} \tan(x)\,dx$$

$\rightarrow \tan(\pi/2) = +\infty$

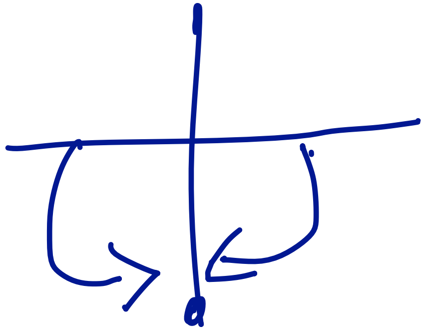
$\rightarrow \tan(3\pi/2) = -\infty$

A definite integral is improper if:

- The function has a vertical asymptote at *x*=*a*, *x*=*b*, or at some point *c* in the interval (*a*,*b*).
- One or both of the limits of integration are infinite (positive or negative infinity).

$$\int_{-\infty}^{\infty} f(x)\,dx, \quad \int_1^{\infty} g(x)\,dx$$

$$\lim_{x \to \frac{3\pi}{2}} \tan(x) = \lim_{x \to \frac{3\pi}{2}} \frac{\sin(x)}{\cos(x)}$$



$$= \lim_{x \to \frac{-\pi}{2}} \tan(x)$$

Which of the following integral(s) is (are) improper? Why / which case?

$\checkmark$ 1) $\displaystyle\int_0^{\frac{\pi}{4}} \tan(2x)\,dx$

$\checkmark$ 2) $\displaystyle\int_{-1}^{1} \frac{x-3}{x^2-2x-3}\,dx$

$\times$ 3) $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\,dx$

$\times$ 4) $\displaystyle\int_0^{3} \frac{x-2}{x^2-6x+8}\,dx = I$

$\tan\left(\frac{2\cdot\pi}{4}\right) = +\infty$

$\hookrightarrow \lim\limits_{x\to\frac{\pi}{4}} \tan(2x) = +\infty$

(2) $d(x) = x^2 - 2x - 3$

$= (x-3)(x+1)$

$d(-1) = 0 \rightarrow$ vertical asympt.

at $x=a=-1$

(3) NO: $\cos(x)$ does not have any vertical asymptotes from 0 to $\pi/2$

(4) $d(x) = x^2 - 6x + 8$

$= (x-2)(x-4)$

is $d(x)$ zero anywhere for $x \in [0,3]$? $x=2$

$$I = \int_0^3 \frac{(x-2)}{d(x)}\,dx = \int_0^3 \frac{dx}{x-4}$$

Convergence of an Integral

- If an improper integral evaluates to a **finite number**, we say it ***converges***.

- If the integral evaluates to $\pm\infty$ or to, typo $\cancel{\infty - \infty}$, we say the integral ***diverges***.

Case 1: At Least One Infinite Limit

Redefine the integral into one of the following.

$$(i)\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

$$(ii)\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

$$(iii)\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$$

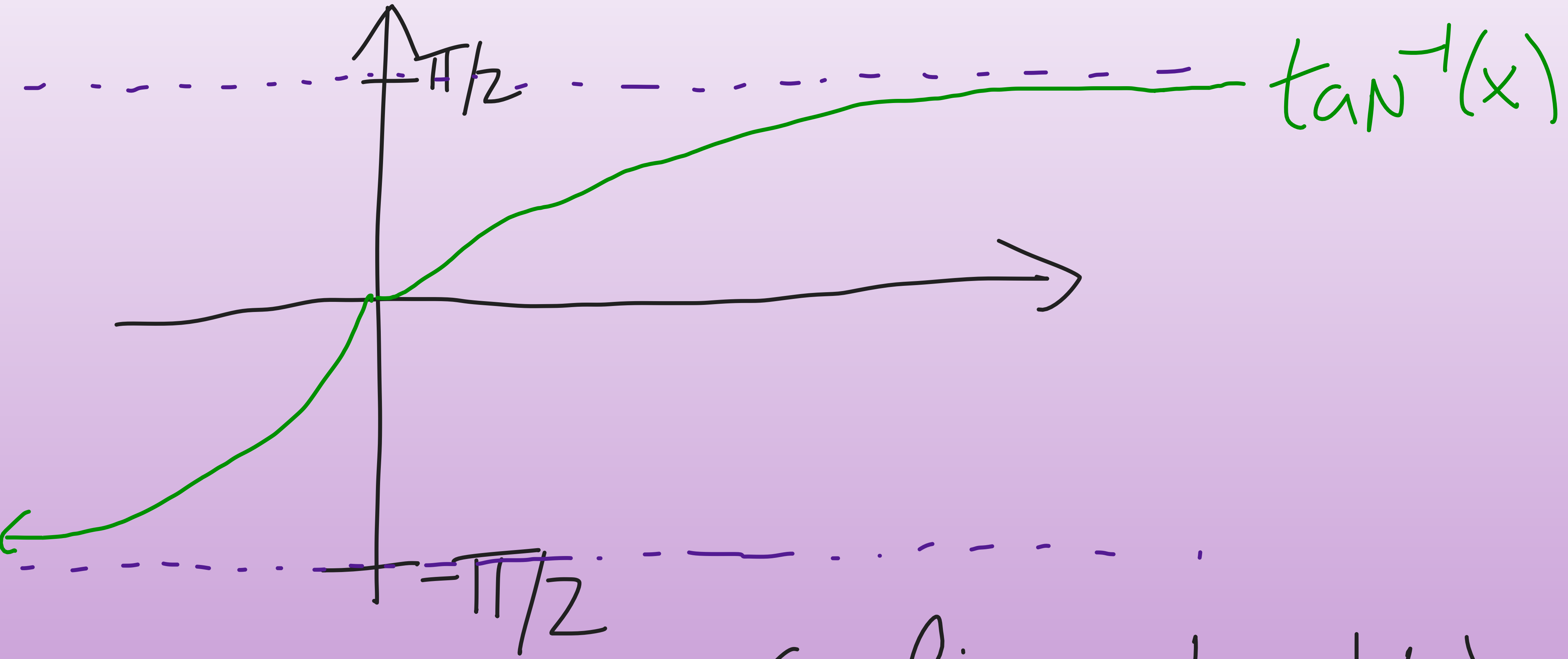
$$\text{and now use parts } (i) \text{ and } (ii).$$

Example 1.1: Evaluate the integral: $\int_{-\infty}^{0} \frac{dx}{1+x^2} = I$

$\rightarrow$ $1+x^2 \neq 0$ for any $x \in (-\infty, 0]$, so no vertical asymptotes

$\rightarrow$ $I = \tan^{-1}(0) - \lim_{b \to -\infty} \tan^{-1}(b)$

0 since $\tan(0) = 0$



so $\lim\limits_{b \to -\infty} \tan^{-1}(b) = -\frac{\pi}{2}$

$I = \pi/2$

Example 1.2: Evaluate the integral: $\int_0^\infty x^3 e^{-x^2}\,dx = I$

→ No vertical asymptotes of $x^3 e^{-x^2}$ for $x \in [0, +\infty)$

→ first evaluate the indefinite integral:

u-sub: $u = x^2$, $du = 2x\,dx$

$$\int x^3 e^{-x^2}\,dx = \frac{1}{2}\cdot\int u e^{-u}\,du$$

→ Now use IBP

$$u = u \qquad dv = e^{-u}\,du$$

$$du = du \qquad v = -e^{-u}$$

$$\left(\text{IBP: } \int u\,dv = uv - \int v\,du\right)$$

$$\frac{1}{2}\cdot\int u e^{-u}\,du = -\frac{u}{2}e^{-u} + \frac{1}{2}\int e^{-u}\,du$$

$$= -\frac{u}{2}e^{-u} - \frac{e^{-u}}{2} + C$$

$$= -\frac{x^2}{2} \cdot e^{-x^2} - \frac{e^{-x^2}}{2} + C$$

$\rightarrow$ To evaluate I:

$$I = \lim_{b\to\infty} \left(-\frac{b^2 \cdot e^{-b^2}}{2} - \frac{e^{-b^2}}{2} \right)$$

$$- \left(0 - \frac{e^0}{2} \right)$$

$$= 0 + 0 - \left(-\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Steps:

1. find points that make the integral improper
2. rewrite as limits
3. find antiderivative of the indefinite integral
4. apply FTC with limits

Case 2: f(c)→∞ Between *a* and *b*

- Case 2 occurs when f has a vertical asymptote on the interval [*a*,*b*].
- Redefine the integral into one of the following.

$$(i)\ \text{If } f(a)\ \cancel{\text{DNE}} \to \pm\infty, \text{ then}: \int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$$

$$(ii)\ \text{If } f(b)\ \cancel{\text{DNE}}\ \pm\infty, \text{ then}: \int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$$

$$(iii)\ \text{If } f(c)\ \cancel{\text{DNE}}\ \pm\infty, \text{ where } a<c<b, \text{ then}:$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

and now use parts (i) and (ii).

Example 2.1: Evaluate the integral: $\int_{\frac{\pi}{2}}^{\pi} \tan(x)dx = I$

$\rightarrow$ problem point at $x = a = \pi/2$

$$I = \lim_{a \to \frac{\pi}{2}^+} \int_a^{\pi} \tan x \, dx$$

$$= \lim_{a \to \frac{\pi}{2}^+} \left(\ln|\sec x| \right)\Big|_a^{\pi}$$

$$\ln|\sec(\pi)| = \ln(1) = 0$$

$$|\cos(\tfrac{\pi}{2})| = 0$$

$$\lim_{a\to\frac{\pi}{2}} |\sec(a)| = +\infty$$

so, $\lim_{a\to\pi/2} \ln|\sec(a)| = +\infty$

so, $I = -\infty$ (diverges)

Example 2.2: Evaluate the integral: $\int_{-1}^{32} \frac{dx}{x^5} = I$

(Sketch the solution)

$\rightarrow$ vertical asymptote of $\frac{1}{x^5}$ at $x = 0$

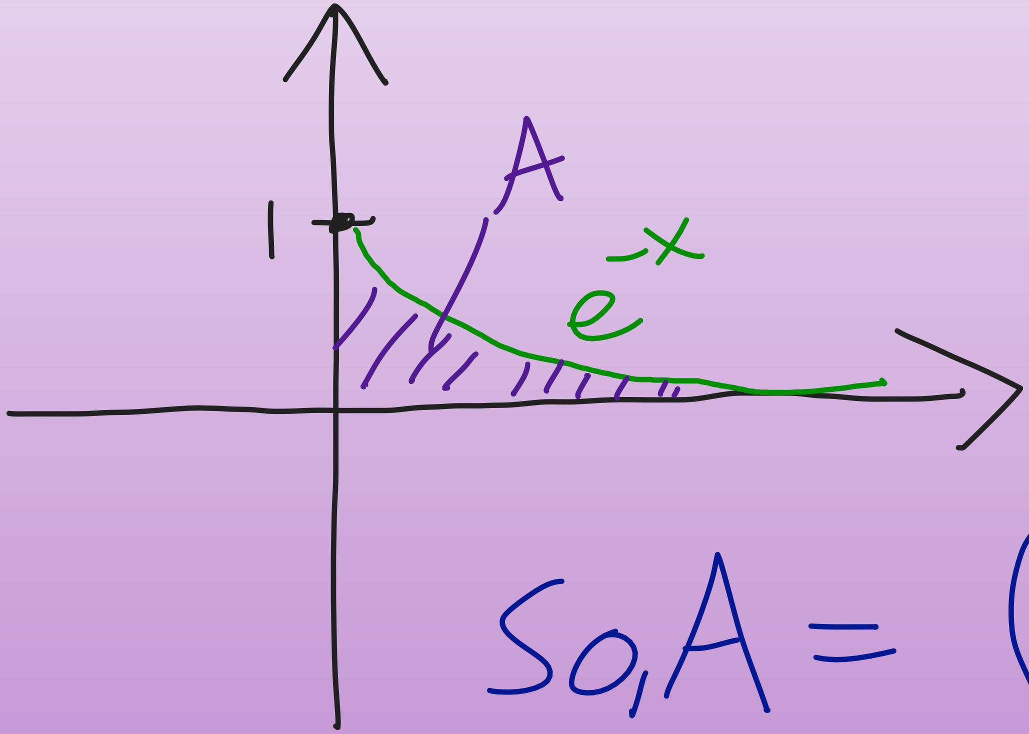
$$\rightarrow I = \int_{-1}^{0} \frac{dx}{x^5} + \int_{0}^{32} \frac{dx}{x^5}$$

$$= \lim_{c \to 0^-} \int_{-1}^{c} \frac{dx}{x^5} + \lim_{c \to 0^+} \int_{c}^{32} \frac{dx}{x^5}$$

(diverges: work this out on your own)

Example 3: Find the area of the region bounded by

$y = e^{-x}$, the x-axis, and $x \geq 0$



$e^{-x} > 0$ always

So, $A = \int_0^\infty e^{-x}\,dx$

$$A = \lim_{b \to \infty} -e^{-x} \Big|_0^b$$

$$= \lim_{b \to \infty} -e^{-b} - \left(-e^{0}\right)$$

$$= 0 + 1$$

Bonus Problems on Improper Integrals

Evaluate each of the next integrals (*if time permits*). → Sketch the solutions

- 1 $\displaystyle\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,dx$
- 2 $\displaystyle\int_0^\infty \frac{e^{-\frac{1}{2x}}}{x^2}\,dx$
- 3 $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$
- 4 $\displaystyle\int_1^e \frac{dx}{x\sqrt{\ln(x)}}$ (converges)
- 5 $\displaystyle\int_e^\infty \frac{dx}{x\sqrt{\ln(x)}}$ (diverges)

1 Problem point at $x = 0$.

$$I_1 = \lim_{c \to 0^+} \int_c^1 \frac{\ln(x)\,dx}{\sqrt{x}}$$

② $I_2 = \int_0^\infty \frac{e^{-\frac{1}{2x}}}{x^2}\,dx$

$$= \lim_{c \to 0^+} \int_c^\infty \frac{e^{-\frac{1}{2x}}}{x^2}\,dx$$

$\rightarrow$ still have to apply a limit at the upper bound of int.

$\rightarrow$ to eval the indefinite integral?

u-sub! $u = \frac{1}{2x}$, $du = -\frac{dx}{2x^2}$

③ $$I_3 = \int_0^{\infty} \frac{e^x}{e^{2x}+3}\,dx$$

$$= \lim_{b \to \infty} \int_0^b \frac{e^x}{e^{2x}+3}\,dx$$

→ To evaluate the indefinite integral:

u-sub: $u = e^x$, $du = e^x\,dx$

$$\int \frac{du}{u^2+3} = \frac{1}{3}\int \frac{du}{\left(\frac{u}{\sqrt{3}}\right)^2+1}$$

4 $\displaystyle\int_1^e \frac{dx}{x\sqrt{\ln(x)}}$ (converges) $\rightarrow$ problem point at $x = e$

5 $\displaystyle\int_e^\infty \frac{dx}{x\sqrt{\ln(x)}}$ (diverges)

$\downarrow$

upper limit of int is $+\infty$